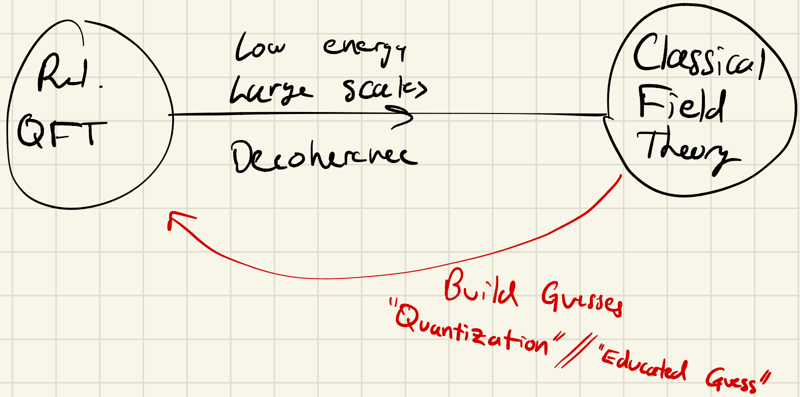
Lecture 2: Classical Field Theory

Content: Classical Field Theory, Quantization of Classical Fields, Klein-Gordon Equations

Last day: Failure of Single-Particle RQM, Poincaré groups, unitary operators, Subgroups



What is a Classical Field?

A field ϕ is a quantity defined at every point in a manifold $\mathcal{M}$ (called spacetime).

i.e. density, spin, charge, ...

For QFT, $\mathcal{M} \equiv \mathcal{M}_{1,3} = \mathbb{R}^1 \times \mathbb{R}^3$

1 time dimension
3 spatial dimensions

A Field is a function

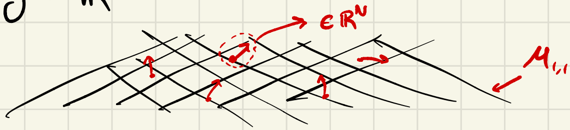
$\phi : \mathcal{M} \longrightarrow S$

$S \equiv$ "target space"

Example: (1) $S = \mathbb{R}^1$ Scalar field

- Higgs boson
- charge, magnetization densities

(2) $S = \mathbb{R}^N$ Vector field



- Pions
- (Electromagnetic field)

(3) $S = S^2$ Surface of a sphere

- "σ-model"
- Quantum magnets

(4) S = Torus $(S^1 \times S^1) \longrightarrow$ Chern-Simons Theorem

Lie Group $\rightarrow$

When $S = \mathbb{R}^N$, we write $\phi(x) \in \mathbb{R}^N$. $\rightarrow x \in M_{1,3}$

Equivalently, we describe $\phi_a(x)$ as a list; ie. $a \in 1, \ldots, N$ of N scalar fields.

Dynamics of a Classical Field:

Consider the dynamics generated by Lagrangians: a system of several fields $\phi_a(x)$. The index 'a' labels

Scalar Field (For now...)

also particle types (charge, ...).

Restrict attention to fields whose classical dynamics is obtained by means of a variational principle applied to an action functional.

Our action functionals involve Lagrange Densities $\mathcal{L}$ which is a function of ϕ_a, $\partial_\mu \phi_a$, $\partial_\nu \partial_\mu \phi_a$, ... etc

measurable set

$$S(\Omega) = \int_\Omega \mathcal{L}(\phi_a, \partial_\mu \phi_a)\, d^4x \qquad (2.1)$$

$d^4x = dx_0 dx_1 dx_2 dx_3$

where $\Omega \subseteq \mathcal{M}_{1,3}$ is a region in spacetime and

Typically $\Omega = \mathcal{M}_{1,3}$.

we assume $\mathcal{L}$ is <u>not</u> a function of $\partial_\nu \partial_\mu \phi_a$ and higher derivatives.

Extract Equations of Motion: Suppose S is Stationary under Variations

$$\phi_a(x) \longrightarrow \phi_a(x) + \delta\phi_a(x) \qquad (2.2)$$

which vanish on $\partial\Omega$. $\rightarrow$ Boundary of Ω

Thus:

$$\delta S(\Omega) = \int_\Omega d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi_a} \delta\phi_a + \underline{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta(\partial_\mu \phi_a)} \right\} = 0 \qquad (2.3)$$

↓ Integrate the 2^{nd} term by parts

$$= \int_\Omega d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) \right\} \delta\phi_a + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta\phi_a \right) \Bigg|_\Omega \qquad (2.4)$$

$$= \int_\Omega d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) \right\} \delta\phi_a + \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta\phi_a \Bigg|_{\partial\Omega}}_{=0\,;\ \delta\phi_a = 0 \text{ on } \partial\Omega} \qquad (2.5)$$

Since $\delta\phi_a$ arbitrary, we find

$$\boxed{\frac{\partial \mathcal{L}}{\partial \phi_a} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right)}_{\text{E-L Equations}} \qquad (2.6)$$

Which are the Euler-Lagrange Equations for ϕ_a.

Benefit of Action Principle:

- Easy to design
- Equations of motion with given symmetries.

Example: Klein-Gordon field

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2 \qquad (2.7)$$

where $\quad \dot{\phi} = \partial_0 \phi \;;\; (\nabla \phi)_j = \partial_j \phi$

$$\Longrightarrow \quad \mathcal{L} = \frac{1}{2} \underbrace{(\partial_\mu \phi)^2}_{=(\partial_\mu \phi)(\partial^\mu \phi)} - \frac{1}{2} m^2 \phi^2 \qquad (2.8)$$

By (2.6), we find that

- $\dfrac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi$
- $\dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi$

$$\frac{1}{2} \frac{\partial(\partial_\kappa \phi \partial^\kappa \phi)}{\partial(\partial_\mu \phi)} = \frac{1}{2} \frac{\partial(\partial_\kappa \phi\, \eta^{\kappa\lambda} \partial_\lambda \phi)}{\partial(\partial_\mu \phi)}$$
$$= \frac{1}{2}\left(\delta^\mu_\kappa\, \eta^{\kappa\lambda} \partial_\lambda \phi + \partial_\kappa \phi\, \eta^{\kappa\lambda} \delta^\mu_\lambda\right)$$
$$= \frac{1}{2}\left(\partial^\mu \phi + \partial^\mu \phi\right)$$
$$= \partial^\mu \phi$$

$$\partial_\mu (\partial^\mu \phi) = -m^2 \phi$$

Thus $$(\partial_\mu \partial^\mu + m^2)\phi = 0 \qquad (2.10)$$

or $$\boxed{(\Box + m^2)\phi = 0} \qquad (2.11)$$

Klein-Gordon Equation

Hamiltonian Formalism:

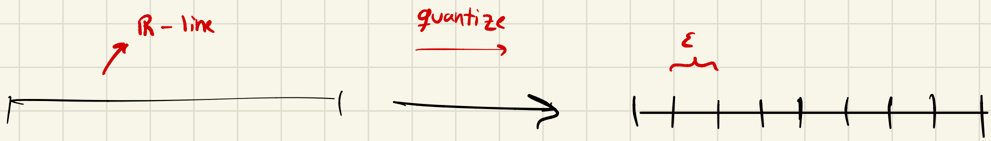
To guess a quantum theory w/ classical limit determined by $\mathcal{L}$, we want to find conjugate variables and then impose canonical commutation relations.

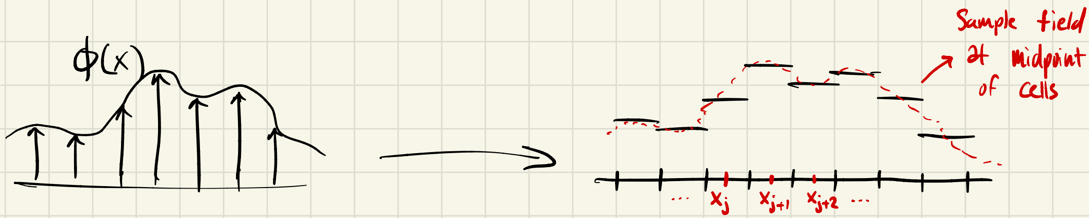
ANS: if $\phi_a(x)$ is "canonical position", define the momentum density via

$$\Pi_a(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_a} \qquad (2.12)$$

Restrict to $\mathcal{M}_{1,1}$ and discretize the field theory.

$\longrightarrow$ Discretize space into regular lattice of spacing 'ε'.





Generalized coordinates

$$q_j(t) \equiv \phi_a(t, x_j) \qquad (j \in \mathbb{Z}) \qquad (2.13)$$

Equations of motion: discretize action & apply variational principle.

$$\partial_\mu \phi_a(t,x) \overset{?}{\longrightarrow}$$

Taylor's Theorem

$$\partial_x \phi(t, x_j) \cong \frac{\phi(t, x_j + \varepsilon) - \phi(t, x_j)}{\varepsilon}$$

$$= \frac{q^a_{j+\varepsilon} - q^a_j}{\varepsilon} \qquad (2.14)$$

and $\dot{q}_j \equiv \partial_t \phi(t, x_j)$

Lagrangian: $L(t) = \int_{\mathbb{R}} d^{(1)}x \, \mathcal{L}(\phi_a, \partial_\mu \phi_a)$

$$= L(q^a_j(t), \dot{q}^a_j(t))$$

Integral turns into discrete sum over space

$$\cong \sum_{j'} \delta x_{j'} \mathcal{L}(\phi_a(t, x_j), \partial_\mu \phi_a(t, x_j)) \qquad (2.15)$$

Conjugate momenta $p_j(t) = \frac{\partial L}{\partial \dot{q}^a_j}$

$$= \sum_{j'} \delta x_{j'} \frac{\partial \mathcal{L}}{\partial \dot{q}^a_j}$$

This must also be a density !!

$$P^a_j(t) \equiv \pi^a(t, x_j)\, \delta x_j \qquad (2.16)$$

Example: $\mathcal{L} = \frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi$

$$= \frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}(\partial_x \phi)^2 \qquad (2.17)$$

→ Discretize

$$L(t) = \frac{1}{2} \sum_{j=-\infty}^{\infty} \varepsilon \left(\frac{dq_j}{dt}\right)^2 - \frac{(q_{j+\varepsilon} - q_j)^2}{\varepsilon} \qquad (2.18)$$

Then the conjugate momentum is

$$p_j(t) = \varepsilon \frac{dq_j}{dt} = \varepsilon\, \pi(t, x_j) \qquad (2.19)$$

Then, taking the infinitesimal limit as $\varepsilon \to 0$, we obtain that

$$\begin{aligned} q_j(t) &\longrightarrow \phi(t,x) \\ p_j(t) &\longrightarrow \dot{\phi}(t,x) \end{aligned} \qquad (2.20)$$

Hamiltonian Density:

For a discrete system:

$$H = \sum_j p_j^a \dot{q}_j^a - L \qquad (2.21)$$

$$= \sum_j \delta x_j \left\{ \pi_a(t, x_j)\, \dot{\phi}_a(t, x_j) - \mathcal{L}_j \right\} \qquad (2.22)$$

As $\delta x_j \to 0$, $\sum_j \to \int d^3x$

This motivates the definition for Hamiltonian Density

$$H = \int d^3x \, \mathcal{H}(x) \qquad (2.23)$$

where

$$\boxed{\mathcal{H}(x) \equiv \pi_a(t,x)\,\dot{\phi}_a(t,x) - \mathcal{L}(\phi_a, \partial_\mu \phi_a)} \qquad (2.24)$$

Hamilton Density

Example: Klein-Gordon field:

$$\mathcal{H}(x) = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2$$

Hamiton density

(Energy density for <u>time-independent</u> fields)